

Polished version, arXiv:1009.5375.

## A NEW SERIES FOR $\pi^3$ AND RELATED CONGRUENCES

ZHI-WEI SUN

Department of Mathematics, Nanjing University  
Nanjing 210093, People's Republic of China  
zwsun@nju.edu.cn  
<http://math.nju.edu.cn/~zwsun>

ABSTRACT. Let  $H_n^{(2)}$  denote the second-order harmonic number  $\sum_{0 < k \leq n} 1/k^2$  for  $n = 0, 1, 2, \dots$ . In this paper we obtain the following new identity:

$$\sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}} = \frac{\pi^3}{48}.$$

We explain how we found the series and develop related congruences involving Bernoulli or Euler numbers, e.g., it is shown that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^{(2)} \equiv -E_{p-3} \pmod{p}$$

for any prime  $p > 3$ , where  $E_0, E_1, E_2, \dots$  are Euler numbers. Motivated by the Amdeberhan-Zeilberger identity  $\sum_{k=1}^{\infty} (21k-8)/(k^3 \binom{2k}{k}^3) = \pi^2/6$ , we also establish the congruence

$$\sum_{k=1}^{(p-1)/2} \frac{21k-8}{k^3 \binom{2k}{k}^3} \equiv (-1)^{(p+1)/2} 4E_{p-3} \pmod{p}$$

for each prime  $p > 3$ .

---

2010 *Mathematics Subject Classification*. Primary 11B65, 11Y60; Secondary 05A10, 05A19, 11A07, 11B68, 33E99.

*Keywords*. Series for  $\pi^3$ , central binomial coefficients, congruences modulo prime powers, Bernoulli and Euler numbers.

Supported by the National Natural Science Foundation (grant 11171140) of China.

## 1. INTRODUCTION

Series with summations related to  $\pi$  have a long history. Leibniz and Euler got the famous identities

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

respectively. Though there exist many series for  $\pi$  and  $\pi^2$  (see, e.g., R. Matsumoto [Ma]), there are very few interesting series for  $\pi^3$ . The only well-known series for  $\pi^3$  is the following one:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}. \quad (1.1)$$

The author [Su2] suggested that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} = \frac{7\pi^3}{216} \quad (1.2)$$

and a readable WZ proof of (1.2) can be found in [HP].

Recall that harmonic numbers are those rational numbers

$$H_n := \sum_{0 < k \leq n} \frac{1}{k} \quad (n = 0, 1, 2, \dots),$$

and *harmonic numbers of the second order* are defined by

$$H_n^{(2)} := \sum_{0 < k \leq n} \frac{1}{k^2} \quad (n = 0, 1, 2, \dots).$$

Now we give our first result which appears to be new and curious.

**Theorem 1.1.** *We have the following new identity:*

$$\sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}} = \frac{\pi^3}{48}. \quad (1.3)$$

*Remark 1.1.* The author noted that **Mathematica 7** could not evaluate the series in (1.3).

By Stirling's formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad (n \rightarrow +\infty)$$

and thus

$$\binom{2k}{k} \sim \frac{4^k}{\sqrt{k\pi}} \quad (k \rightarrow +\infty).$$

Note also that  $H_n^{(2)} \rightarrow \zeta(2) = \pi^2/6$  as  $n \rightarrow \infty$ . Therefore

$$\frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}} \sim \frac{\zeta(2)\sqrt{\pi}}{2^k \sqrt{k}} \quad (k \rightarrow +\infty).$$

So the series in (1.3) converges much faster than the series in (1.1) (but slower than the series in (1.2)). Using **Mathematica 7** we found that for  $n \geq 500$  we have

$$\left| \frac{s_n}{\pi^3/48} - 1 \right| < \frac{1}{10^{150}}$$

where  $s_n := \sum_{k=1}^n 2^k H_{k-1}^{(2)} / (k \binom{2k}{k})$ .

The reader may wonder how the author discovered (1.3) which gives a series for  $\pi^3$  of a new type. Now we present some explanations.

Let  $p$  be an odd prime. In [Su3] and [Su4] the author proved the congruences

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3} \quad (1.4)$$

and

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}}{(-2)^k} \equiv (-1)^{(p-1)/2} 2^{p-1} \pmod{p^3} \quad (1.5)$$

respectively, where  $E_0, E_1, E_2, \dots$  are Euler numbers given by  $E_0 = 1$  and the recursion

$$\sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = 0 \quad (n = 1, 2, 3, \dots).$$

For  $k = 0, \dots, p-1$ , clearly we have

$$\begin{aligned} \binom{p-1}{k} (-1)^k &= \prod_{0 < j \leq k} \left(1 - \frac{p}{j}\right) \\ &\equiv 1 - pH_k + \frac{p^2}{2} \sum_{0 < i < j \leq k} \frac{2}{ij} = 1 - pH_k + \frac{p^2}{2} (H_k^2 - H_k^{(2)}) \pmod{p^3}. \end{aligned}$$

So, in view of (1.4) and (1.5), it is natural to investigate

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^{(2)} \pmod{p}, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^2 \pmod{p}, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k \pmod{p^2}.$$

This led the author to obtain the following result.

**Theorem 1.2.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^{(2)} \equiv -E_{p-3} \pmod{p}. \quad (1.6)$$

*Remark 1.2.* Let  $p$  be an odd prime. We are also able to show that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^2 \equiv \left( \frac{-1}{p} \right) \frac{q_p(2)^2}{2} - \frac{E_{p-3}}{2} \pmod{p}, \quad (1.7)$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k \equiv \left( \frac{-1}{p} \right) \frac{H_{(p-1)/2}}{2} - pE_{p-3} \pmod{p^2}, \quad (1.8)$$

where  $\left( \frac{\cdot}{p} \right)$  denotes the Legendre symbol, and  $q_p(2)$  stands for the Fermat quotient  $(2^{p-1} - 1)/p$ . Recall that in 1938 Lehmer [L] proved the congruence

$$H_{(p-1)/2} \equiv -2q_p(2) + p q_p(2)^2 \pmod{p^2}. \quad (1.9)$$

In view of certain correspondence between series for the zeta function or powers of  $\pi$  and congruences involving Bernoulli or Euler numbers revealed in the authors' papers [Su2] and [Su3], the congruence (1.6) suggests that we should consider the series  $\sum_{k=0}^{\infty} \binom{2k}{k} H_k^{(2)} / 2^k$ . Since this series diverges, we should seek for certain transformation. Let  $p$  be an odd prime. It is easy to see that

$$\frac{1}{p} \binom{2(p-k)}{p-k} \equiv -\frac{2}{k \binom{2k}{k}} \pmod{p} \quad \text{for } k = 1, \dots, \frac{p-1}{2}.$$

(Cf. [Su2, Lemma 2.1] and [T].) Thus, if  $p > 3$  then

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^{(2)} &\equiv \sum_{k=1}^{(p-1)/2} \frac{k \binom{2k}{k}}{k 2^k} H_k^{(2)} \equiv \sum_{k=1}^{(p-1)/2} \left( \frac{H_k^{(2)}}{k 2^k} \cdot \frac{-2p}{\binom{2(p-k)}{p-k}} \right) \\ &\equiv \sum_{p/2 < k < p} \left( \frac{H_{p-k}^{(2)}}{(p-k) 2^{p-k}} \cdot \frac{-2p}{\binom{2k}{k}} \right) \\ &\equiv -p \sum_{p/2 < k < p} \frac{2^k H_{p-k}^{(2)}}{k \binom{2k}{k}} \equiv -p \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}} \pmod{p} \end{aligned}$$

since  $2^p \equiv 2 \pmod{p}$  and

$$-H_{p-k}^{(2)} \equiv H_{p-1}^{(2)} - H_{p-k}^{(2)} \equiv H_{k-1}^{(2)} \pmod{p}.$$

Therefore the congruence in (1.6) is equivalent to

$$p \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}} \equiv E_{p-3} \pmod{p}. \quad (1.6')$$

Motivated by (1.6') the author found (1.3).

Now we state our third theorem which is close to Theorem 1.2.

**Theorem 1.3.** *Let  $p$  be an odd prime. If  $p > 3$ , then*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{4^k} H_k \equiv 2 - 2p + 4p^2 q_p(2) \pmod{p^3}. \quad (1.11)$$

We also have

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} H_k^{(2)} \equiv -4q_p(2) \pmod{p} \quad (1.12)$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k4^k} H_k^{(2)} \equiv \frac{B_{p-3}}{2} \pmod{p}, \quad (1.13)$$

where  $B_0, B_1, B_2, \dots$  are Bernoulli numbers.

In 1997 T. Amdeberhan and D. Zeilberger [AZ] obtained that

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3 \binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

We are able to establish the following result related to the Amdeberhan-Zeilberger series.

**Theorem 1.4.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} (21k+8) \binom{2k}{k}^3 \equiv 8p + (-1)^{(p-1)/2} 32p^3 E_{p-3} \pmod{p^4} \quad (1.14)$$

and hence

$$\sum_{k=1}^{(p-1)/2} \frac{21k-8}{k^3 \binom{2k}{k}^3} \equiv (-1)^{(p+1)/2} 4E_{p-3} \pmod{p}. \quad (1.15)$$

*Remark 1.3.* In [Su3] the author showed that

$$\sum_{k=0}^{p-1} (21k+8) \binom{2k}{k}^3 \equiv 8p + 16p^4 B_{p-3} \pmod{p^5} \quad (1.16)$$

for any odd prime  $p$ . However, (1.14) is much more sophisticated than this congruence involving  $B_{p-3}$ .

The next section is devoted to the proof of Theorem 1.1. We are going to show Theorems 1.2–1.3 and Theorem 1.4 in Sections 3 and 4 respectively. Section 5 contains some conjectures of the author for further research.

## 2. PROOF OF THEOREM 1.1

Set

$$S := \sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}}.$$

Then

$$S = \sum_{k=0}^{\infty} \frac{2^{k+1} H_k^{(2)}}{(k+1) \binom{2k+2}{k+1}} = \sum_{k=0}^{\infty} \frac{2^k H_k^{(2)}}{(k+1) \binom{2k+1}{k}} = \sum_{k=0}^{\infty} \frac{2^k H_k^{(2)} \Gamma^2(k+1)}{\Gamma(2k+2)}.$$

Recall the well-known fact that

$$B(a, b) := \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{for any } a, b > 0.$$

So we have

$$\begin{aligned} S &= \sum_{k=0}^{\infty} 2^k H_k^{(2)} \int_0^1 x^k (1-x)^k dx = \sum_{k=0}^{\infty} \frac{H_k^{(2)}}{2^k} \int_0^1 (1-(2x-1)^2)^k dx \\ &= \sum_{k=0}^{\infty} \frac{H_k^{(2)}}{2^{k+1}} \int_{-1}^1 (1-t^2)^k dt = \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{2^k} \int_0^1 (1-t^2)^k dt. \end{aligned}$$

Observe that if  $0 \leq t \leq 1$  then

$$\begin{aligned} \sum_{k=1}^{\infty} H_k^{(2)} \left( \frac{1-t^2}{2} \right)^k &= \sum_{k=1}^{\infty} \sum_{j=1}^k \frac{1}{j^2} \left( \frac{1-t^2}{2} \right)^k = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{k=j}^{\infty} \left( \frac{1-t^2}{2} \right)^k \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2} \left( \frac{1-t^2}{2} \right)^j \frac{1}{1 - (1-t^2)/2} \\ &= \frac{2}{1+t^2} \text{Li}_2 \left( \frac{1-t^2}{2} \right), \end{aligned}$$

where the dilogarithm  $\text{Li}_2(x)$  is given by

$$\text{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (|x| < 1).$$

Therefore

$$\frac{S}{2} = \int_0^1 \frac{1}{1+t^2} \text{Li}_2 \left( \frac{1-t^2}{2} \right) dt = \int_0^1 \text{Li}_2 \left( \frac{1-t^2}{2} \right) (\arctan t)' dt.$$

Note that

$$\text{Li}_2'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\frac{\log(1-x)}{x}$$

and hence

$$\frac{d}{dt} \text{Li}_2\left(\frac{1-t^2}{2}\right) = -\frac{\log(1-(1-t^2)/2)}{(1-t^2)/2} \times (-t) = \frac{2t}{1-t^2} \log \frac{1+t^2}{2}.$$

Thus, using integration by parts we obtain

$$\begin{aligned} \frac{S}{2} &= \text{Li}_2\left(\frac{1-t^2}{2}\right) \arctan t \Big|_{t=0}^1 - \int_0^1 (\arctan t) \frac{2t}{1-t^2} \log \frac{1+t^2}{2} dt \\ &= \int_0^1 (\arctan t) \left( \frac{1}{1+t} - \frac{1}{1-t} \right) \log \frac{1+t^2}{2} dt \\ &= \int_0^1 \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt - \int_0^{-1} \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt \\ &= \int_{-1}^1 \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt. \end{aligned}$$

Finally, inputting the Mathematica command

$$\text{Integrate}[\text{ArcTan}[t] \text{Log}[(1+t^2)/2]/(1+t), \{t, -1, 1\}]$$

we then obtain from **Mathematica 7** that

$$\int_{-1}^1 \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt = \frac{\pi^3}{96}.$$

Thus  $S = \pi^3/48$  as desired. We are done.

### 3. PROOFS OF THEOREMS 1.2 AND 1.3

We first state some basic facts which will be used very often. For any prime  $p > 3$  we have

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}$$

since  $\sum_{j=1}^{p-1} (2j)^{-2} \equiv \sum_{k=1}^{p-1} k^{-2} \pmod{p}$ . If  $p$  is an odd prime, then

$$\binom{(p-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \quad \text{for each } k = 0, \dots, p-1. \quad (3.1)$$

For any  $n = 0, 1, 2, \dots$  we have the identity

$$\sum_{k=0}^n (-1)^k \binom{x}{k} = (-1)^n \binom{x-1}{n} \quad (3.2)$$

which can be found in [G, (1.5)].

**Lemma 3.1.** *For any positive integer  $n$ , we have the identities*

$$\sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1}}{k} = H_n \quad (3.3)$$

and

$$\sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1}}{k} H_k = H_n^{(2)}. \quad (3.4)$$

*Proof.* (3.3) and (3.4) follow from [G, (1.45)] and an identity of V. Hernández [He] respectively. Below we give a simple proof of (3.4). In view of the binomial inversion formula (cf. (5.48) of [GKP, pp.192-193]), (3.4) holds for all  $n = 1, 2, 3, \dots$  if and only if for any positive integer  $n$  we have

$$\sum_{k=1}^n \binom{n}{k} (-1)^k H_k^{(2)} = -\frac{H_n}{n}. \quad (3.4')$$

In fact, in view of (3.2) and (3.3), we get

$$\begin{aligned} \sum_{k=1}^n \binom{n}{k} (-1)^k \sum_{j=1}^k \frac{1}{j^2} &= \sum_{j=1}^n \frac{1}{j^2} \left( \sum_{k=0}^n \binom{n}{k} (-1)^k - \sum_{k=0}^{j-1} \binom{n}{k} (-1)^k \right) \\ &= \sum_{j=1}^n \frac{(-1)^j}{j^2} \binom{n-1}{j-1} = \frac{1}{n} \sum_{j=1}^n \frac{(-1)^j}{j} \binom{n}{j} = -\frac{H_n}{n} \end{aligned}$$

and hence (3.4') holds.  $\square$

**Lemma 3.2.** *Let  $p = 2n + 1$  be an odd prime and let  $m$  be an integer with  $m \not\equiv 0, 4 \pmod{p}$ . Then*

$$\sum_{k=1}^n \frac{\binom{2k}{k}}{m^k} H_k^{(2)} \equiv - \left( \frac{m(m-4)}{p} \right) \sum_{k=1}^n \frac{\binom{2k}{k} H_k}{k(4-m)^k} \pmod{p}. \quad (3.5)$$

*In particular,*

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{2^k} H_k^{(2)} \equiv - \left( \frac{-1}{p} \right) \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k} H_k}{k 2^k} \pmod{p}. \quad (3.6)$$

*Proof.* Clearly it suffices to prove (3.5).



In view of (3.4), we have

$$\begin{aligned}
\sum_{k=1}^n \binom{n}{k} \left(-\frac{4}{m}\right)^k H_k^{(2)} &= \sum_{k=1}^n \frac{\binom{2k}{k}}{m^k} \sum_{j=1}^k \binom{k}{j} \frac{(-1)^{j-1}}{j} H_j \\
&= \sum_{j=1}^n \frac{(-1)^{j-1}}{j} H_j \sum_{k=j}^n \binom{n}{k} \binom{k}{j} \left(-\frac{4}{m}\right)^k \\
&= \sum_{j=1}^n \frac{(-1)^{j-1}}{j} H_j \binom{n}{j} \sum_{k=j}^n \binom{n-j}{k-j} \left(-\frac{4}{m}\right)^k \\
&= \sum_{j=1}^n \binom{n}{j} \frac{(-1)^{j-1}}{j} H_j \left(-\frac{4}{m}\right)^j \left(1 - \frac{4}{m}\right)^{n-j} \\
&= -\frac{1}{m^n} \sum_{j=1}^n \binom{n}{j} \frac{4^j H_j}{j} (m-4)^{n-j}
\end{aligned}$$

So, with the help of (3.1) we obtain

$$\sum_{k=1}^n \frac{\binom{2k}{k}}{m^k} H_k^{(2)} \equiv -\left(\frac{m(m-4)}{p}\right) \sum_{j=1}^n \frac{\binom{2j}{j} (-1)^j H_j}{j(m-4)^j} \pmod{p}.$$

This proves (3.5). We are done.  $\square$

**Lemma 3.3.** *Let  $n$  be any positive integer. Then*

$$\sum_{k=1}^n \binom{n}{k} \frac{(-2)^k}{k} H_k = -2 \sum_{\substack{k=1 \\ 2 \nmid k}}^n \frac{H_n - H_{n-k}}{k}. \quad (3.7)$$

*Proof.* Let  $S_n$  denote the left-hand side of (3.7). Observe that

$$\begin{aligned}
S_n &= \sum_{k=1}^n \binom{n}{k} \frac{(-2)^k}{k} \sum_{j=1}^k \int_0^1 x^{j-1} dx = \int_0^1 \sum_{k=1}^n \binom{n}{k} \frac{(-2)^k}{k} \cdot \frac{x^k - 1}{x-1} dx \\
&= \int_0^1 \int_0^1 \sum_{k=0}^n \binom{n}{k} \frac{(-2x)^k - (-2)^k}{x-1} y^{k-1} dy dx \\
&= \int_0^1 \int_0^1 \frac{(1-2xy)^n - (1-2y)^n}{(x-1)y} dy dx \\
&= -2 \int_0^1 \int_0^1 \sum_{k=1}^n (1-2xy)^{k-1} (1-2y)^{n-k} dx dy
\end{aligned}$$

Clearly,

$$\int_0^1 (1-2xy)^{k-1} dx = \frac{(1-2xy)^k}{-2ky} \Big|_{x=0}^1 = \frac{(1-2y)^k - 1}{k(1-2y-1)} = \frac{1}{k} \sum_{j=1}^k (1-2y)^{j-1}.$$

Therefore

$$\begin{aligned} S_n &= -2 \int_0^1 \sum_{k=1}^n \frac{1}{k} \sum_{j=1}^k (1-2y)^{n-k+j-1} dy = \sum_{1 \leq j \leq k \leq n} \frac{(1-2y)^{n-k+j}}{k(n-k+j)} \Big|_{y=0}^1 \\ &= \sum_{1 \leq j \leq k \leq n} \frac{(-1)^{n-k+j} - 1}{k(n-k+j)} = \sum_{i=1}^n \frac{(-1)^i - 1}{i} \sum_{j=1}^i \frac{1}{n+j-i} \\ &= -2 \sum_{\substack{i=1 \\ 2 \nmid i}}^n \frac{1}{i} (H_n - H_{n-i}). \end{aligned}$$

This completes the proof of (3.7).  $\square$

**Lemma 3.4.** *Let  $p > 3$  be a prime. Then*

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3} \quad (3.8)$$

*Remark 3.1.* (3.8) is a famous congruence of Morley [Mo].

**Lemma 3.5.** *Let  $p > 3$  be a prime. Then*

$$\sum_{\substack{k=1 \\ 2 \nmid k}}^{(p-1)/2} \frac{H_k}{k} \equiv \frac{3}{4} q_p(2)^2 - \left( \frac{-1}{p} \right) \frac{E_{p-3}}{2} \pmod{p} \quad (3.9)$$

and

$$\sum_{\substack{k=1 \\ 2 \mid k}}^{(p-1)/2} \frac{H_k}{k} \equiv \frac{5}{4} q_p(2)^2 + \left( \frac{-1}{p} \right) \frac{E_{p-3}}{2} \pmod{p}. \quad (3.10)$$

*Proof.* Set  $n = (p-1)/2$ . Clearly it suffices to show that

$$\sum_{k=1}^n \frac{H_k}{k} \equiv 2q_p(2)^2 \pmod{p} \quad (3.11)$$

and

$$\sum_{k=1}^n \frac{(-1)^k}{k} H_k \equiv \frac{q_p(2)^2}{2} + \left( \frac{-1}{p} \right) E_{p-3} \pmod{p}. \quad (3.12)$$

Let  $\delta \in \{0, 1\}$ . For  $r = 0, \dots, p-1$  we obviously have

$$(-1)^r \binom{p-1}{r} = \prod_{0 < s \leq r} \left(1 - \frac{p}{s}\right) \equiv 1 - pH_r \pmod{p^2}.$$

Thus

$$\begin{aligned} p \sum_{k=1}^n \frac{(-1)^{\delta k}}{k} H_{k-1} &\equiv \sum_{k=1}^n \frac{(-1)^{\delta k}}{k} \left(1 - (-1)^{k-1} \binom{p-1}{k-1}\right) \\ &= \sum_{k=1}^n \frac{(-1)^{\delta k}}{k} + \frac{1}{p} \sum_{k=1}^n (-1)^{(\delta+1)k} \binom{p}{k} \pmod{p^2} \end{aligned}$$

and hence

$$p \sum_{k=1}^n \frac{(-1)^{\delta k}}{k} H_k \equiv \sum_{k=1}^n (-1)^{\delta k} \left(\frac{1}{k} + \frac{p}{k^2}\right) + \frac{1}{p} \sum_{k=1}^n (-1)^{(\delta+1)k} \binom{p}{k} \pmod{p^2}. \quad (3.13)$$

Putting  $\delta = 0$  in (3.13) and recalling (3.2) and the congruence  $\sum_{k=1}^n 1/k^2 \equiv 0 \pmod{p}$ , we get

$$p \sum_{k=1}^n \frac{H_k}{k} \equiv H_n + \frac{(-1)^n \binom{p-1}{n} - 1}{p} \pmod{p}.$$

With helps of (1.9) and (3.8), we have

$$p \sum_{k=1}^n \frac{H_k}{k} \equiv -2q_p(2) + pq_p(2)^2 + \frac{(1 + pq_p(2))^2 - 1}{p} \equiv 2pq_p(2)^2 \pmod{p^2}$$

which yields (3.11). Taking  $\delta = 1$  in (3.13) and using the congruence  $\sum_{k=1}^n 1/k^2 \equiv 0 \pmod{p}$ , we obtain

$$\begin{aligned} p \sum_{k=1}^n \frac{(-1)^k}{k} H_k &\equiv \sum_{k=1}^n \frac{(-1)^k + 1}{k} + p \sum_{k=1}^n \frac{(-1)^k + 1}{k^2} - H_n + \frac{2^p - 2}{2p} \\ &\equiv H_{\lfloor p/4 \rfloor} + \frac{p}{2} \sum_{j=1}^{\lfloor p/4 \rfloor} \frac{1}{j^2} - H_n + q_p(2) \pmod{p^2}. \end{aligned}$$

Let's recall (1.9) and note that

$$\sum_{k=1}^{\lfloor p/4 \rfloor} \frac{1}{k^2} \equiv 4 \left(\frac{-1}{p}\right) E_{p-3} \pmod{p} \quad (3.14)$$

and

$$H_{[p/4]} \equiv -3q_p(2) + \frac{3}{2}p q_p(2)^2 - \left(\frac{-1}{p}\right) pE_{p-3} \pmod{p^2}$$

by Lehmer [L, (20)] and [S2, Corollary 3.3] respectively. Therefore,

$$\begin{aligned} p \sum_{k=1}^n \frac{(-1)^k}{k} H_k &\equiv -3q_p(2) + \frac{3}{2}p q_p(2)^2 - \left(\frac{-1}{p}\right) pE_{p-3} \\ &\quad + 2p \left(\frac{-1}{p}\right) E_{p-3} + (2q_p(2) - p q_p(2)^2) \\ &\equiv \frac{p}{2} q_p(2)^2 + \left(\frac{-1}{p}\right) pE_{p-3} \pmod{p^2} \end{aligned}$$

and hence (3.12) holds. We are done.  $\square$

**Lemma 3.6.** *Let  $p$  be an odd prime. Then*

$$\sum_{\substack{k=1 \\ 4|k-2}}^{p-1} \frac{H_k}{k} \equiv \frac{3}{16} q_p(2)^2 \pmod{p}. \quad (3.15)$$

If  $p > 3$ , then we also have

$$\sum_{\substack{k=1 \\ 4|k}}^{p-1} \frac{H_k}{k} \equiv \frac{5}{16} q_p(2)^2 \pmod{p}. \quad (3.16)$$

*Proof.* As  $H_{p-k} \equiv H_{k-1} \pmod{p}$  for  $k = 1, \dots, p-1$ , we have

$$\begin{aligned} p \sum_{\substack{k=1 \\ 4|k-2}}^{p-1} \frac{H_k}{k} &= p \sum_{\substack{k=1 \\ 4|k-p+2}}^{p-1} \frac{H_{p-k}}{p-k} \\ &\equiv - \sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{pH_{k-1}}{k} \equiv \sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{(-1)^{k-1} \binom{p-1}{k-1} - 1}{k} \\ &= \sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{1}{k} \binom{p-1}{k-1} - \sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{1}{k} \pmod{p^2} \end{aligned}$$

Note that

$$2 \sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{1}{k} \binom{p-1}{k-1} = q_p(2) - \frac{\left(\frac{2}{p}\right) 2^{(p-1)/2} - 1}{p} = \frac{2^{p-1} - \left(\frac{2}{p}\right) 2^{(p-1)/2}}{p}$$

by [Su1, Corollary 3.1] and

$$\sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{1}{k} \equiv \frac{q_p(2)}{4} - \frac{p}{8} q_p(2)^2 \pmod{p^2}$$

by [S2, Corollary 3.1]. Therefore

$$\begin{aligned} p \sum_{\substack{k=1 \\ 4|k-2}}^{p-1} \frac{H_k}{k} &\equiv \frac{2^{p-1} - \left(\frac{2}{p}\right) 2^{(p-1)/2}}{2p} - \frac{2^{p-1} - 1}{4p} + \frac{p}{8} q_p(2)^2 \\ &= p \left( \frac{\left(\frac{2}{p}\right) 2^{(p-1)/2} - 1}{2p} \right)^2 + \frac{p}{8} q_p(2)^2 \\ &\equiv p \left( \frac{2^{p-1} - 1}{4p} \right)^2 + \frac{p}{8} q_p(2)^2 = \frac{3}{16} p q_p(2)^2 \pmod{p^2} \end{aligned}$$

and hence (3.15) follows. When  $p > 3$  we can prove (3.16) in a similar way.  $\square$

*Proof of Theorem 1.2.* Set  $n = (p-1)/2$ . In view of (3.1) and (3.6), it suffices to show

$$\sum_{k=1}^n \binom{n}{k} \frac{(-2)^k}{k} H_k \equiv \left( \frac{-1}{p} \right) E_{p-3} \pmod{p}.$$

For each  $k = 1, \dots, n$ , evidently

$$H_n - H_{n-k} = \sum_{j=0}^{k-1} \frac{1}{n-j} \equiv -2 \sum_{j=0}^{k-1} \frac{1}{2j+1} = -2 \left( H_{2k} - \frac{H_k}{2} \right) \pmod{p}.$$

Thus, in light of (3.7), (3.15) and (3.9), we have

$$\begin{aligned} \sum_{k=1}^n \binom{n}{k} \frac{(-2)^k}{k} H_k &\equiv 4 \sum_{\substack{k=1 \\ 2 \nmid k}}^n \left( \frac{H_{2k}}{k} - \frac{H_k}{2k} \right) = 8 \sum_{\substack{j=1 \\ 4 \nmid j-2}}^{p-1} \frac{H_j}{j} - 2 \sum_{\substack{k=1 \\ 2 \nmid k}}^n \frac{H_k}{k} \\ &\equiv \frac{3}{2} q_p(2)^2 - \frac{3}{2} q_p(2)^2 + \left( \frac{-1}{p} \right) E_{p-3} \pmod{p} \end{aligned}$$

as desired. This concludes the proof.  $\square$

**Lemma 3.7.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv B_{p-3} \pmod{p} \quad (3.17)$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \equiv -2B_{p-3} \pmod{p}. \quad (3.18)$$

*Remark 3.2.* (3.17) appeared as [ST, (5.4)], and (3.18) follows from [S1, Corollary 5.2(b)].

**Lemma 3.8.** *For any positive integer  $m$  and nonnegative integer  $n$  we have*

$$\sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k+m} = \frac{1}{m \binom{m+n}{m}}. \quad (3.19)$$

*Remark 3.3.* (3.19) can be found in [G, (1.43)].

*Proof of Theorem 1.3.* Observe that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{4^k} H_k &= \sum_{k=1}^{p-1} \binom{-1/2}{k} (-1)^k \sum_{j=1}^k \frac{1}{j} \\ &= \sum_{j=1}^{p-1} \frac{1}{j} \left( \sum_{k=0}^{p-1} \binom{-1/2}{k} (-1)^k - \sum_{k=0}^{j-1} \binom{-1/2}{k} (-1)^k \right). \end{aligned}$$

Applying (3.2) we get

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{4^k} H_k &= \sum_{j=1}^{p-1} \left( (-1)^{p-1} \binom{-1/2-1}{p-1} - (-1)^{j-1} \binom{-1/2-1}{j-1} \right) \\ &= \binom{-3/2}{p-1} H_{p-1} - 2 \sum_{j=1}^{p-1} (-1)^j \frac{-1/2}{j} \binom{-3/2}{j-1} \\ &= \binom{-3/2}{p-1} H_{p-1} - 2 \left( \sum_{j=0}^{p-1} (-1)^j \binom{-1/2}{j} - 1 \right) \\ &= \binom{-3/2}{p-1} H_{p-1} - 2 \binom{-1/2-1}{p-1} + 2. \end{aligned}$$

Now assume  $p > 3$ . Note that

$$\binom{-3/2}{p-1} = \frac{p}{-1/2} \binom{-1/2}{p} = -2p \frac{\binom{2p}{p}}{(-4)^p} = p \frac{\binom{2p-1}{p-1}}{4^{p-1}} \equiv \frac{p}{4^{p-1}} \pmod{p^4}$$

since  $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$  by Wolstenholme's theorem (see, e.g., [HT]). As  $H_{p-1} \equiv 0 \pmod{p^2}$ , by the above we have

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{4^k} H_k &\equiv 2 - 2 \binom{-3/2}{p-1} \equiv 2 - \frac{2p}{(1 + p q_p(2))^2} \\ &\equiv 2 - 2p(1 - p q_p(2))^2 = 2 - 2p + 4p^2 q_p(2) \pmod{p^3}. \end{aligned}$$

So (1.11) holds.

Below we write  $p = 2n + 1$ . Combining (3.1), (3.4') and (1.9), we get

$$\sum_{k=0}^n \frac{\binom{2k}{k}}{4^k} H_k^{(2)} \equiv -\frac{H_{(p-1)/2}}{(p-1)/2} \equiv 2H_{(p-1)/2} \equiv -4q_p(2) \pmod{p}.$$

This proves (1.12).

In view of (3.4) and (3.19), we have

$$\begin{aligned} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^k}{k} H_k^{(2)} &= \sum_{k=1}^n \binom{n}{k} \frac{(-1)^k}{k} \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} \frac{H_j}{j} \\ &= - \sum_{j=1}^n \frac{H_j}{j} \binom{n}{j} \sum_{k=j}^n \frac{(-1)^{k-j}}{k} \binom{n-j}{k-j} \\ &= - \sum_{j=1}^n \frac{H_j}{j} \binom{n}{j} \frac{1}{j \binom{n}{j}} = - \sum_{j=1}^n \frac{H_j}{j^2}. \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k}{k^2} &= \sum_{k=1}^n \left( \frac{H_k}{k^2} + \frac{H_{p-k}}{(p-k)^2} \right) \\ &\equiv \sum_{k=1}^n \left( \frac{H_k}{k^2} + \frac{H_{k-1}}{k^2} \right) = 2 \sum_{k=1}^n \frac{H_k}{k^2} - \sum_{k=1}^n \frac{1}{k^3} \pmod{p}. \end{aligned}$$

Therefore

$$\sum_{k=1}^n \frac{\binom{2k}{k}}{4^k} H_k^{(2)} \equiv - \sum_{k=1}^n \frac{H_k}{k^2} \equiv -\frac{1}{2} \left( \sum_{k=1}^{p-1} \frac{H_k}{k^2} + \sum_{k=1}^n \frac{1}{k^3} \right) \pmod{p}.$$

Now applying Lemma 3.7 we immediately get the desired (1.13).

The proof of Theorem 1.3 is now complete.  $\square$

## 4. PROOF OF THEOREM 1.4

**Lemma 4.1.** *For any positive integer  $n$ , we have the following identities:*

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k = 2(-1)^n H_n, \quad (4.1)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k^{(2)} = 2(-1)^{n-1} \sum_{k=1}^n \frac{(-1)^k}{k^2}. \quad (4.2)$$

*Remark 4.1.* (4.1) and (4.2) can be found in [OS] and [Pr].

**Lemma 4.2.** *Let  $p = 2n + 1$  be an odd prime. Then*

$$\begin{aligned} \frac{\binom{n+k}{k}}{\binom{2k}{k}/4^k} &\equiv 1 + p \sum_{j=1}^k \frac{1}{2j-1} + \frac{p^2}{2} \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 \\ &\quad - \frac{p^2}{2} \sum_{j=1}^k \frac{1}{(2j-1)^2} \pmod{p^3} \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \frac{\binom{n}{k}}{\binom{2k}{k}/(-4)^k} &\equiv 1 - p \sum_{j=1}^k \frac{1}{2j-1} + \frac{p^2}{2} \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 \\ &\quad - \frac{p^2}{2} \sum_{j=1}^k \frac{1}{(2j-1)^2} \pmod{p^3}. \end{aligned} \quad (4.4)$$

In particular,

$$\binom{n}{k} \binom{n+k}{k} (-1)^k \equiv \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}. \quad (4.5)$$

*Proof.* Observe that

$$\begin{aligned} \frac{\binom{n+k}{k}}{\binom{2k}{k}/4^k} &= \prod_{j=1}^k \frac{(n+j)/j}{(2j-1)/(2j)} = \prod_{j=1}^k \left( 1 + \frac{p}{2j-1} \right) \\ &\equiv 1 + p \sum_{j=1}^k \frac{1}{2j-1} + \frac{p^2}{2} S_k \pmod{p^3} \end{aligned}$$

where

$$S_k := 2 \sum_{1 \leq i < j \leq k} \frac{1}{(2i-1)(2j-1)} = \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 - \sum_{j=1}^k \frac{1}{(2j-1)^2}.$$



This proves (4.3). Similarly,

$$\frac{(-1)^k \binom{n}{k}}{\binom{2k}{k}/4^k} = \prod_{j=1}^k \left(1 - \frac{p}{2j-1}\right) \equiv 1 - p \sum_{j=1}^k \frac{1}{2j-1} + p^2 S_k \pmod{p^3}$$

and hence (4.4) holds. Clearly (4.5) follows from (4.3) and (4.4). We are done.  $\square$

**Lemma 4.3.** *For any nonnegative integer  $n$  we have*

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad (4.6)$$

and

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{(-4)^k} = \frac{\binom{2n}{n}}{4^n}. \quad (4.7)$$

*Remark 4.2.* As  $\binom{n}{k} = \binom{n}{n-k}$  and  $\binom{2k}{k}/(-4)^k = \binom{-1/2}{k}$  for all  $k = 0, \dots, n$ , both (4.6) and (4.7) are special cases of the Chu-Vandermonde identity  $\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}$  (cf. [G, (3.1)] or (5.22) of [GKP, p. 169]).  $\square$

**Lemma 4.5.** *Let  $n$  be any positive integer. Then*

$$t_n := \frac{1}{4n \binom{2n}{n}} \sum_{k=0}^{n-1} (21k + 8) \binom{2k}{k}^3$$

coincides with

$$t'_n := \sum_{k=0}^{n-1} \binom{n+k-1}{k}^2.$$

*Remark 4.3.* In Feb. 2010, the author conjectured that  $t_n$  is always an integer and later this was confirmed by Kasper Andersen by getting  $t_n = t'_n$  via the Zeilberger algorithm (cf. [Su3, Lemma 4.1]).

Now we are ready to prove the following auxiliary result.

**Theorem 4.1.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{H_k}{16^k} \equiv 2 \left( \frac{-1}{p} \right) H_{(p-1)/2} \pmod{p^2}, \quad (4.8)$$

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k}^2 \frac{H_k^{(2)}}{16^k} \equiv -4E_{p-3} \pmod{p}, \quad (4.9)$$

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k}^2 \frac{H_k}{k16^k} \equiv 4 \left( \frac{-1}{p} \right) E_{p-3} \pmod{p}, \quad (4.10)$$

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{H_{2k}}{16^k} \equiv \left( \frac{-1}{p} \right) \frac{3}{2} H_{(p-1)/2} + pE_{p-3} \pmod{p^2}. \quad (4.11)$$

*Proof.* Set  $n = (p-1)/2$ . In view of (4.5), (4.1) implies (4.8), and (4.2) yields that

$$\sum_{k=0}^n \binom{2k}{k}^2 \frac{H_k^{(2)}}{16^k} \equiv 2(-1)^{n-1} \sum_{k=1}^n \frac{(-1)^k}{k^2} \pmod{p^2}.$$

Since  $\sum_{k=1}^n 1/k^2 \equiv 0 \pmod{p}$ , we have

$$\sum_{k=1}^n \frac{(-1)^k}{k^2} \equiv \sum_{k=1}^n \frac{(-1)^k + 1}{k^2} = \frac{1}{2} \sum_{j=1}^{\lfloor p/4 \rfloor} \frac{1}{j^2} \equiv 2(-1)^n E_{p-3} \pmod{p}$$

by applying (3.14) in the last step. Now it is clear that (4.9) holds.

Now we deduce (4.10). With helps of (3.4) and the Chu-Vandermonde identity, we get

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k^{(2)} \\ &= \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} (-1)^k \sum_{j=1}^k \binom{k}{j} \frac{(-1)^{j-1}}{j} H_j \\ &= \sum_{j=1}^n \binom{n}{j} \frac{(-1)^{j-1}}{j} H_j \sum_{k=j}^n \binom{n+k}{k} (-1)^k \binom{n-j}{k-j} \\ &= \sum_{j=1}^n \binom{n}{j} \frac{(-1)^{j-1}}{j} H_j \sum_{k=0}^n \binom{-n-1}{k} \binom{n-j}{n-k} \\ &= \sum_{j=1}^n \binom{n}{j} \frac{(-1)^{j-1}}{j} H_j \binom{-j-1}{n} = (-1)^{n-1} \sum_{j=1}^n \binom{n}{j} \binom{n+j}{j} \frac{(-1)^j}{j} H_j. \end{aligned}$$

Thus, by applying (4.5) we obtain (4.10) from (4.9).

Since

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k}^2 H_{2k}^{(2)} &= \sum_{k=0}^n \binom{n}{k}^2 H_{2(n-k)} = \sum_{k=0}^n \binom{n}{k}^2 H_{p-1-2k}^{(2)} \\ &\equiv - \sum_{k=0}^n \binom{n}{k}^2 H_{2k}^{(2)} \pmod{p}, \end{aligned}$$

by (3.1) we have

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} H_{2k}^{(2)} \equiv \sum_{k=0}^n \binom{n}{k}^2 H_{2k}^{(2)} \equiv 0 \pmod{p}$$

and hence

$$\begin{aligned} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \sum_{j=1}^k \frac{1}{(2j-1)^2} &= \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \left( H_{2k}^{(2)} - \frac{H_k^{(2)}}{4} \right) \\ &\equiv - \frac{1}{4} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} \pmod{p}. \end{aligned}$$

Thus (4.9) implies that

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \sum_{j=1}^k \frac{1}{(2j-1)^2} \equiv E_{p-3} \pmod{p}. \quad (4.12)$$

Observe that

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k = \sum_{k=0}^n \binom{n}{n-k} \binom{-n-1}{k} = \binom{-1}{n} = (-1)^n$$

by the Chu-Vandermonde identity. Combining this with (4.3) and (4.4), we get

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \left( 1 - p^2 \sum_{j=1}^k \frac{1}{(2j-1)^2} \right) \equiv (-1)^n \pmod{p^3}.$$

From this and (4.12) we see that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}, \quad (4.13)$$

which was first established in [Su3] via the math. software **Sigma**.

By (4.6) and (4.7), we have

$$\begin{aligned} \left(1 - \frac{2}{4^n}\right) \binom{2n}{n} &= \sum_{k=0}^n \binom{n}{k} \left( \binom{n}{k} - \frac{2 \binom{2k}{k}}{(-4)^k} \right) \\ &= \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \cdot \frac{\binom{n}{k}}{\binom{2k}{k}/(-4)^k} \left( \frac{\binom{n}{k}}{\binom{2k}{k}/(-4)^k} - 2 \right). \end{aligned}$$

Combining this with (4.4) we get

$$\left(1 - \frac{2}{4^n}\right) \binom{2n}{n} \equiv \sum_{k=1}^n \frac{\binom{2k}{k}^2}{16^k} \left( p^2 \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 - 1 \right) \pmod{p^3}.$$

By Morley's congruence (3.8),

$$\left(1 - \frac{2}{4^n}\right) \binom{2n}{n} + (-1)^n = (-1)^n (4^{2n} - 2 \cdot 4^n + 1) = (-1)^n p^2 q_p(2)^2 \pmod{p^3}.$$

Thus, in light of (4.13) we obtain

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 \equiv E_{p-3} + \left( \frac{-1}{p} \right) q_p(2)^2 \pmod{p}. \quad (4.14)$$

By (4.7), (4.4), (4.12) and (4.14),

$$\begin{aligned} &\frac{\binom{2n}{n}}{4^n} - \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \left( 1 - p \sum_{j=1}^k \frac{1}{2j-1} \right) \\ &\equiv \frac{p^2}{2} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} \left( \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 - \sum_{j=1}^k \frac{1}{(2j-1)^2} \right) \\ &\equiv \frac{p^2}{2} (-1)^n q_p(2)^2 \pmod{p}. \end{aligned}$$

Combining this with (4.13) we obtain

$$\sum_{k=1}^n \frac{\binom{2k}{k}^2}{16^k} \sum_{j=1}^k \frac{1}{2j-1} \equiv (-1)^n \left( -q_p(2) + \frac{p}{2} q_p(2)^2 \right) + p E_{p-3} \pmod{p^2}. \quad (4.15)$$

Therefore, in view of (4.8), we have

$$\begin{aligned} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{16^k} H_{2k} &= \sum_{k=1}^n \frac{\binom{2k}{k}^2}{16^k} \left( \sum_{j=1}^k \frac{1}{2j-1} + \frac{H_k}{2} \right) \\ &\equiv (-1)^n \left( -q_p(2) + \frac{p}{2} q_p(2)^2 \right) + p E_{p-3} + (-1)^n H_n \\ &\equiv (-1)^n \frac{3}{2} H_n + p E_{p-3} \pmod{p^2}. \end{aligned}$$

This proves (4.11).

So far we have completed the proof of Theorem 4.1.

*Proof of Theorem 1.4.* Write  $p = 2n + 1$ . As

$$4(n+1) \binom{2(n+1)}{n+1} = 8p \binom{2n}{n} \equiv 8p(-1)^n 4^{p-1} \pmod{p^3}$$

by Morley's congruence (3.8), and

$$\begin{aligned} 4^{1-p} &= \left( \frac{1}{1 + pq_p(2)} \right)^2 \\ &\equiv (1 - pq_p(2) + p^2 q_p(2)^2)^2 \equiv 1 - 2pq_p(2) + 3p^2 q_p(2)^2 \pmod{p^3} \end{aligned}$$

in view of Lemma 4.5, (1.14) is reduced to

$$\begin{aligned} \sum_{k=0}^n \binom{n+k}{k}^2 &\equiv \frac{4p^2 E_{p-3} + (-1)^n}{4^{p-1}} \\ &\equiv 4p^2 E_{p-3} + (-1)^n (1 - 2pq_p(2) + 3p^2 q_p(2)^2) \pmod{p^3}. \end{aligned} \quad (4.16)$$

For each  $k = 0, \dots, n$ , by (4.3) we have

$$\begin{aligned} &\binom{n+k}{k}^2 \\ &\equiv \frac{\binom{2k}{k}^2}{16^k} \left( 1 + p \sum_{j=1}^k \frac{1}{2j-1} + \frac{p^2}{2} \left( \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 - \sum_{j=1}^k \frac{1}{(2j-1)^2} \right) \right)^2 \\ &\equiv \frac{\binom{2k}{k}^2}{16^k} \left( 1 + 2p \sum_{j=1}^k \frac{1}{2j-1} + p^2 \left( 2 \left( \sum_{j=1}^k \frac{1}{2j-1} \right)^2 - \sum_{j=1}^k \frac{1}{(2j-1)^2} \right) \right) \pmod{p}. \end{aligned}$$

So we can obtain (4.16) by using (4.12)–(4.15).

Now we deduce (1.15). Combining (1.14) and (1.16) we get

$$\sum_{k=(p+1)/2}^{p-1} (21k+8) \binom{2k}{k}^3 \equiv (-1)^{(p+1)/2} 32p^3 E_{p-3} \pmod{p^4},$$

i.e.,

$$\sum_{k=1}^{(p-1)/2} (21(p-k)+8) \frac{\binom{2(p-k)}{p-k}^3}{p^3} \equiv (-1)^{(p+1)/2} 32E_{p-3} \pmod{p}.$$

By [Su3, Lemma 2.1], for each  $k = 1, \dots, (p-1)/2$  we have

$$\frac{\binom{2(p-k)}{p-k}}{p} \equiv \frac{-2}{k \binom{2k}{k}} \pmod{p}.$$

Therefore

$$\sum_{k=1}^{p-1} (-21k + 8) \left( \frac{-2}{k \binom{2k}{k}} \right)^3 \equiv (-1)^{(p+1)/2} 32E_{p-3} \pmod{p},$$

which gives (1.15).

The proof of Theorem 1.4 is now complete.  $\square$

## 5. SOME RELATED CONJECTURES

We first raise the following conjecture similar to (1.6).

**Conjecture 5.1.** *For any prime  $p > 3$  we have*

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k} H_k^{(2)}}{k} &\equiv \frac{2H_{p-1}}{3p^2} + \frac{76}{135} p^2 B_{p-5} \pmod{p^3}, \\ \sum_{k=1}^{p-1} \frac{\binom{2k}{k} H_k^{(2)}}{k 2^k} &\equiv -\frac{3}{16} \cdot \frac{H_{p-1}}{p^2} + \frac{479}{1280} p^2 B_{p-5} \pmod{p^3}, \\ \sum_{k=1}^{p-1} \frac{\binom{2k}{k} H_k^{(2)}}{k 3^k} &\equiv -\frac{8}{9} \cdot \frac{H_{p-1}}{p^2} + \frac{268}{1215} p^2 B_{p-5} \pmod{p^3}. \end{aligned}$$

*Remark 5.1.* It is known that  $H_{p-1}/p^2 \equiv B_{p-3}/3 \pmod{p}$  for any prime  $p > 3$  (see, e.g., [S1]). In contrast with the congruences in Conjecture 5.1, we note that **Mathematica 7** yields

$$\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}}{k^2 \binom{2k}{k}} = \frac{\pi^4}{1944}, \quad \sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k^2 \binom{2k}{k}} = \frac{\pi^4}{384}, \quad \sum_{k=1}^{\infty} \frac{3^k H_{k-1}^{(2)}}{k^2 \binom{2k}{k}} = \frac{2\pi^4}{243}.$$

The following conjecture is close to Theorem 1.3.

**Conjecture 5.2.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k4^k} H_k &\equiv \frac{7}{6} p B_{p-3} \pmod{p^2}, \\ \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k4^k} H_{2k} &\equiv \frac{7}{3} p B_{p-3} \pmod{p^2}, \\ \sum_{k=1}^{p-1} \frac{4^k H_{k-1}}{k^2 \binom{2k}{k}} &\equiv \frac{2}{3} B_{p-3} \pmod{p}, \\ \sum_{k=1}^{(p-1)/2} \frac{4^k H_{2k-1}}{k^2 \binom{2k}{k}} &\equiv \frac{7}{2} B_{p-3} \pmod{p}, \\ \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k4^k} H_k^{(2)} &\equiv -\frac{3}{2} \cdot \frac{H_{p-1}}{p^2} + \frac{7}{80} p^2 B_{p-5} \pmod{p^3}. \end{aligned}$$

Also,

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2 4^k} H_k \equiv \frac{3}{2} B_{p-3} \pmod{p}, \quad \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2 4^k} H_{2k} \equiv \frac{5}{2} B_{p-3} \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2 4^k} \equiv -\frac{H_{(p-1)/2}^2}{2} - \frac{7}{4} \cdot \frac{H_{p-1}}{p} \pmod{p^3} \quad \text{provided } p > 5.$$

*Remark 5.2.* The author ever conjectured that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k4^k} H_{2k} \equiv -2 \left( \frac{-1}{p} \right) E_{p-3} \pmod{p}$$

for any prime  $p > 3$ , this has been confirmed by his former student Hui-Qin Cao. Using **Mathematica 7** the author found that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{4^k H_{k-1}}{k^2 \binom{2k}{k}} &= 7\zeta(3), \quad \sum_{k=1}^{\infty} \frac{4^k H_{2k-1}}{k^2 \binom{2k}{k}} = \frac{21}{2} \zeta(3), \\ \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k4^k} H_k^{(2)} &= \frac{3}{2} \zeta(3), \quad \sum_{k=1}^{\infty} \frac{4^k H_{k-1}^{(2)}}{k^2 \binom{2k}{k}} = \frac{\pi^4}{24}, \quad \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k^2 4^k} = \frac{\pi^2 - 3 \log^2 4}{6}. \end{aligned}$$

Motivated by Theorem 4.1, we present the following conjecture.

**Conjecture 5.3.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} H_{2k}^{(2)} &\equiv B_{p-3} \pmod{p}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} H_{2k}^{(2)} &\equiv -\frac{5}{2} B_{p-3} \pmod{p}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} H_k^{(2)} &\equiv -12 \frac{H_{p-1}}{p^2} - \frac{74}{5} p^2 B_{p-5} \pmod{p^3}, \\ \sum_{p/2 < k < p} \frac{\binom{2k}{k}^2}{k16^k} H_k^{(2)} &\equiv -\frac{31}{2} p^2 B_{p-5} \pmod{p^3}. \end{aligned}$$

and

$$p^2 \sum_{k=1}^{p-1} \frac{16^k H_{k-1}}{k^2 \binom{2k}{k}^2} \equiv 8 \left( \frac{-1}{p} \right) H_{(p-1)/2} + 16p E_{p-3} \pmod{p^2}.$$

## REFERENCES

- [AZ] T. Amdeberhan and D. Zeilberger, *Hypergeometric series acceleration via the WZ method*, Electron. J. Combin. **4** (1997), no. 2, #R3.
- [G] H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., 1972.
- [GKP] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, New York, 1994.
- [HT] C. Helou and G. Terjanian, *On Wolstenholme's theorem and its converse*, J. Number Theory **128** (2008), 475–499.
- [He] V. Hernández, *Solution IV of problem 10490 (a reciprocal summation identity)*, Amer. Math. Monthly **106** (1999), 589–590.
- [HP] K. Hessami Pilehrood and T. Hessami Pilehrood, *Series acceleration formulas for beta values*, Discrete Math. Theor. Comput. Sci. **12** (2010), 223–236.
- [L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. **39** (1938), 350–360.
- [Ma] R. Matsumoto, *A collection of formulae for  $\pi$* , on-line version is available from the website [http://www.pluto.ai.kyutech.ac.jp/plt/matsumoto/pi\\_small](http://www.pluto.ai.kyutech.ac.jp/plt/matsumoto/pi_small).
- [Mo] F. Morley, *Note on the congruence  $2^{4n} \equiv (-1)^n (2n)!/(n!)^2$ , where  $2n+1$  is a prime*, Ann. Math. **9** (1895), 168–170.
- [OS] R. Osburn and C. Schneider, *Gaussian hypergeometric series and supercongruences*, Math. Comp. **78** (2009), 275–292.
- [Pr] H. Prodinger, *Human proofs of identities by Osburn and Schneider*, Integers **8** (2008), #A10, 8pp (electronic).
- [S1] Z. H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), 193–223.
- [S2] Z. H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory **128** (2008), 280–312.
- [Su1] Z. W. Sun, *On the sum  $\sum_{k \equiv r \pmod{m}} \binom{n}{k}$  and related congruences*, Israel J. Math. **128** (2002), 135–156.



- [Su2] Z. W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (2011), 2219–2238.
- [Su3] Z. W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535. <http://arxiv.org/abs/1001.4453>.
- [Su4] Z. W. Sun, *On sums of binomial coefficients modulo  $p^2$* , Colloq. Math., revised. <http://arxiv.org/abs/0910.5667>.
- [ST] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.
- [T] R. Tauraso, *More congruences for central binomial coefficients*, J. Number Theory **130** (2010), 2639–2649.